Concrete Beton, Volume 80, pp 15; 18-19; 21-23, April 1996

A Method for Predicting the Deflection of Two-way Spanning, Edge-supported, Reinforced Concrete Slabs.

Author: A Boting

Structural Design Engineer, Henry Fagan and Partners, Cape Town, South Africa

ABSTRACT:

Reinforced concrete slabs are designed against failure in flexure and shear at the ultimate limit state. Generally, little consideration is given to the serviceability limit state. A concrete slab will spend most of its life at or below its serviceability limit state and should normally never reach its ultimate limit state.

Two of the most important considerations at the serviceability limit state are deflection and crack control. This paper, which is an extract from an MSc thesis completed at the University of Cape Town, concentrates on the short-term maximum deflection of two-way spanning slabs under service loads.

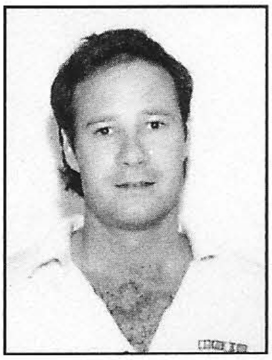
Journal Contact Details:
PO Box 75364
Lynnwood Ridge
Pretoria, 0040
South Africa
+27 12 348 5305

CONCRETE SOCIETY
OF SOUTHERN AFRICA

admin@concretesociety.co.za
www.concretesociety.co.za

A Method for Predicting the Deflection of Two-Way Spannng, Edge-Supported, Reinforced Concrete Slabs

Anthony Boting



Antony Boting was born in Bloemfontein and matriculated at Grey College. He completed both his B.Sc (Civil Engineering) and M.Sc (Civil Engineering) degrees at the University of Cape Town. He is currently employed as a Structural Design Engineer by Henry Fagan and Partners in Cape Town.

SHORT SYNOPSIS

A method, consisting of two computational models, was developed to determine the maximum deflection of two-way spanning, edge supported, reinforced concrete slabs. The first model determined the dispersion of a uniformly distributed service load acting on the slab. The second model determined the maximum deflection of the two orthogonal strips spanning though the region of maximum slab deflection.

INTRODUCTION

Reinforced concrete slabs are designed against failure in flexure and shear at the ultimate limit state. Generally, little consideration is given to the serviceability limit state. A concrete slab will spend most of its life at or below its serviceability limit state and should normally never reach its ultimate limit state. Two of the most important considerations at the serviceability limit state are deflection and crack control. This paper, which is an extract from an M.Sc thesis completed at the University of Cape Town, concentrates on the short-term maximum deflection of two-way spanning slabs under service loads.

Moment of inertia

Deflections are inversely proportional to the moment of inertia of the beam under consideration. The presence of cracks at the serviceability limit state affects the moment of inertia and has a profound effect on the deflection.

A review of Design Codes shows various ways in which deflections can be calculated. Some Codes use a conservative approach and suggest the use of I_{cr}, the cracked moment of inertia, if the tensile capacity of the concrete is exceeded.

Other Codes attempt to include the stiffening effect of the concrete in tension, once cracked. The most notable of these Codes are the American Building Code - ACI 318 M - 83[1] and the Manual on the CEB/FIP Model Code[2].

The American Code makes use of the Branson[3] formula to determine the "effective moment of inertia" I_e of each section along the structural element.

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \quad \text{... eqn 1}$$

where

I_g = moment of inertia of gross concrete section, neglecting reinforcing steel

I_{cr} = moment of inertia of transformed all-concrete cracked section

M_a = moment at the beam section under consideration

M_{cr} = cracking moment of the concrete section

This formula has been verified experimentally and is regarded as being sufficiently accurate for control of deflections.

The Manual on the CEB/FIP Model Code[2] requires that the curvatures of the structural element for both cracked and uncracked states be determined. The uncracked state is referred to as state I and all the concrete and reinforcement are assumed to be active both in tension and compression. The cracked state is referred to as stae II_o and the reinforcement is assumed to be effective in both tension and compression, but the concrete is only effective in compression. The actual curvature $1/r_m$ is then determined by interpolating between the curvature of the uncracked state $1/r_1$ and the curvature of the cracked state $1/r_2$ (figure 1). This relationship can be expressed mathematically as:

$$\frac{1}{r_m} = (1 - \zeta)\frac{1}{r_1} + \zeta\frac{1}{r_2} \quad \text{... eqn 2}$$

Experimentally, the coefficient ζ has been determined as:

$$\zeta = +1 - \beta_1\beta_2\left(\frac{M_r}{M}\right)^2 \quad \text{... eqn 3}$$

$$= 0 \text{ for } M < M_r$$

where

β_1 = a coefficient characterising the bond strength of the reinforcing bars

= 1.0 for high bond bars

= 0.5 for smooth bars

β_2 = a coefficient representing the influence of the duration of application, or of repetition of loading

= 1.0 for first loading

= 0.5 for long-term loads, or for a large number of cycles of load

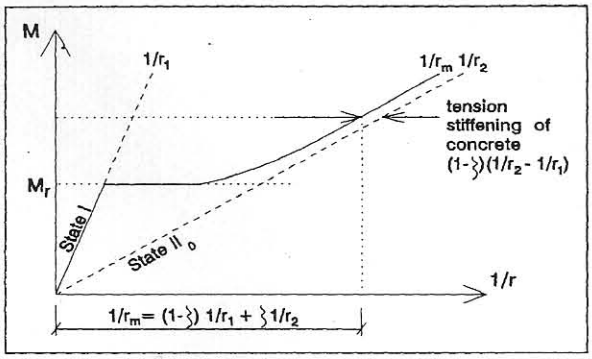


Figure 1 - Instantaneous moment - curvature relationship after Manual on CEB/FIP Model Code (2)

M = moment at the section under consideration

M_r = cracking moment

Three methods for the prediction of deflections are given in the Manual[2]. The most notable is the bilinear method. This is based on the observation that, for the serviceability limit state, the moment-deflection relationship may be approximated by a bilinear relation which represents the overall effect of the moment-curvature relationship described previously (figure 2).

In equation 3, M_r is assumed to be constant over the entire beam element and is taken as equal to the cracking moment capacity of the critical section, which is defined as midspan for a beam. A parameter M_m is defined as the geometric mean of the cracking moment M_r and the maximum total service moment M_d at the critical section and is assumed constant over the beam.

$$M = \sqrt{M_r \cdot M_d} \quad \ldots \; eqn\ 4$$

The methods given in both the ACI Code[1] and the Manual on the CEB/FIP Model Code[2] have a number of simplifications and shortcomings. A method to predict the maximum deflection of two-way spanning slabs needed to be developed. One of the two approaches given above had to be adopted and it was decided to use the approach of the CEB/FIP Model Code. The exact methods given in the Manual on the Code[2] are not used, but the underlying theory is used as a basis for the model that is developed.

COMPUTATIONAL METHOD FOR PREDICTING MAXIMUM DEFLECTIONS

In addition to the factors which affect beam deflections, the deflection of a two-way, edge-supported, rectangular slab panel depends on the boundary conditions at the supports and on the aspect ratio. The load on the slab is resisted not only by orthogonal bending moments, but also by twisting moments and shear.

Two models are developed. One predicts the load dispersion of a two-way spanning slab and effectively produces the equivalent loading on strips or beams spanning in each of the orthogonal directions. The second model predicts the probable maximum deflection of these orthogonal strips or beams spanning through the region of maximum slab deflection.

Model 1 for the equivalent load

The slab is divided up in plan into 5 seperate beam strips in each of the x-and y-directions (figure3). Each strip consists of five zones. The outer two zones are of length L/8, while the three inner zones are of length L/4. There is a node at the centre of each of the three inner zones. Each zone has its own stiffness and unique portion of load that it carries. Deflection formulae are set up in terms of the unknown portions of load

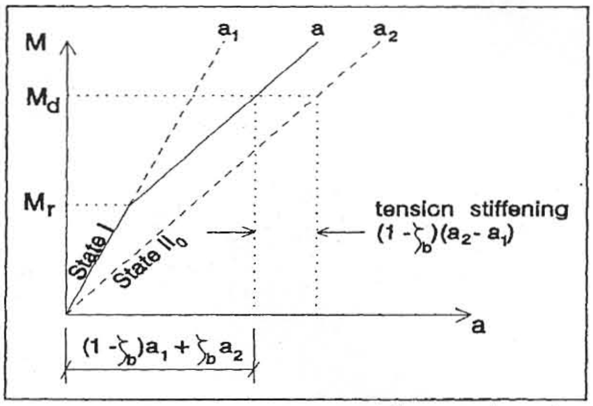


Figure 2 - Instantaneous moment- deflection relationship after Manual on CEB/FIP Model Code [2]

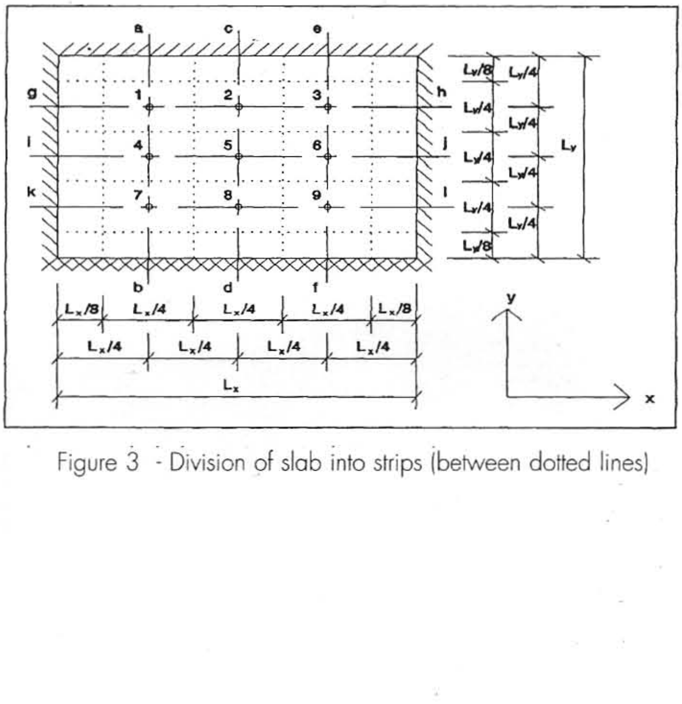


Figure 3 - Division of slab into strips (between dotted lines)

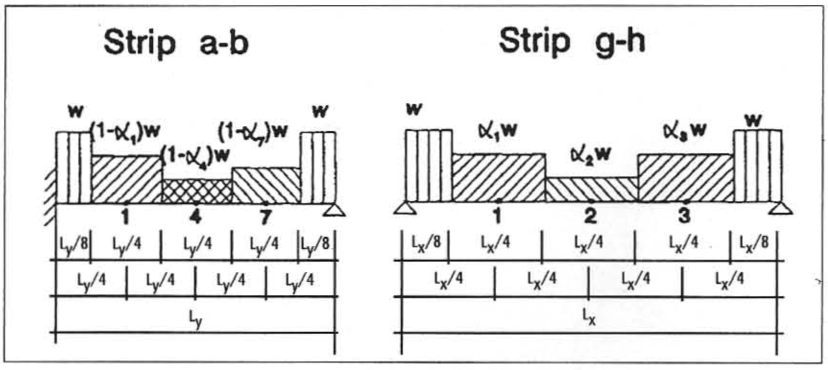


Figure 4 - Example of division of load onto orthogonal strips

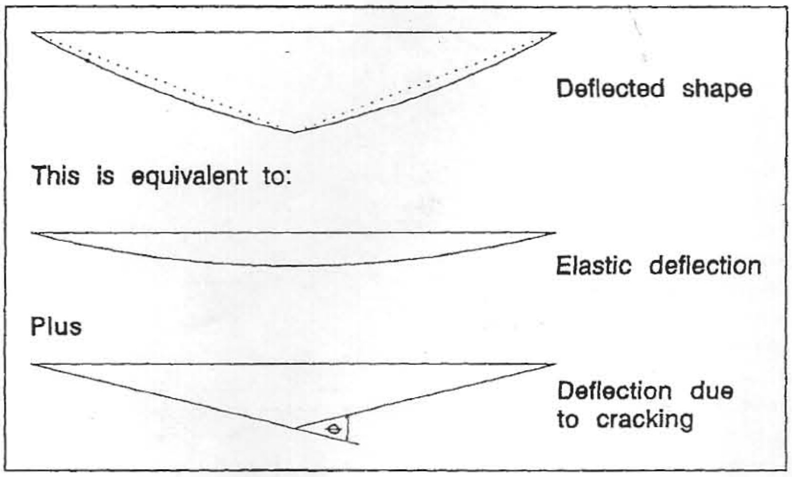


Figure 5 - Deflection Model

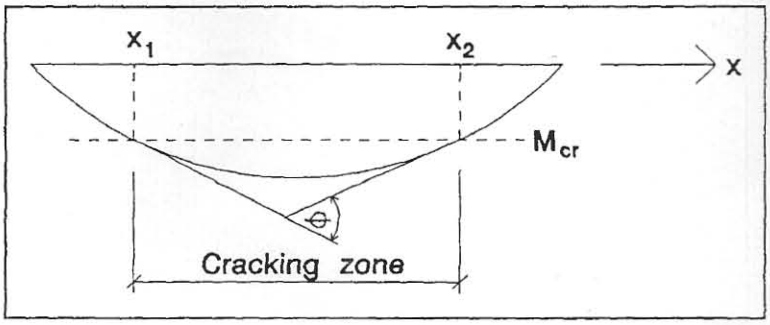


Figure 6 - Cracking Zone

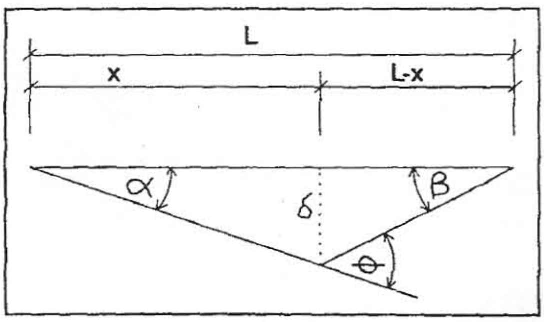


Figure 7 - Calculation of deflection due to hinge at crack

for each strip by integrating the shear force equation three times. These equations are determined for every support condition that can be encountered by a beam.

The deflection at each of the nine nodes (numbered in figure 3) must be the same when determined for the strips in the x- and y- directions. Strips a-b and g-h have the same support conditions as the portion of slab that they represent (figure 4). These two strips intersect at node 1. If the calculated deflections at node 1 for the strips are equal, then strip g-h will carry α_1 of the load and strip a-b will carry $(1-\alpha_1)$ of the load. If the deflections in the orthogonal pairs of strips at each of the nine nodes are equal, then a grid of load dispersions can be determined.
The inner zones carry the portion of load determined as described above, while the two outer zones carry the full load in only one direction. The bending moment can be determined for the loads thus acting on each strip. The cracking moment for each zone is also determined. If the bending moment of a zone exceeds its cracking moment then a new effective stiffness I_{eff} is determined. This stiffness is determined on the following basis:

- If the entire zone is cracked, then the cracked moment of inertia for that zone is determined ignoring concrete in tension.
- If the entire zone is uncracked, then the uncracked moment of inertia for that zone is used.
- If only a portion of the zone is cracked, then a linear interpolation between the cracked and uncracked moments of inertia is used.

These modified stiffnesses are substituted into the deflection equations and a new load dispersion pattern calculated. Once again a new cracked region is determined, and if this differs considerably from the previous one, then the whole procedure is repeated. This iteration procedure carries on until a stable cracking zone for each strip is achieved. This final load dispersion is the one now used to determine deflections. For those slab strips that are statically indeterminate, a check must be made as to whether the plastic moment at the support(s) is exceeded or not. If it is, then the entire iteration procedure is repeated from the start, but the statically indeterminate slab strip is now assumed to be effectively discontinuous with plastic end moment(s). When the bending moment diagram is calculated, the effect of the plastic moment is taken into consideration.

Model 2 for deflections

When determining the maximum deflection, only the strip of slab which will contain this maximum deflection, in each of the x- and y-directions, need be considered. The maximum deflection is then determined for these two orthogonal slab strips.

If the cracking moment is not exceeded, the deflection is determined using elastic formulae and uncracked moments of inertia.

If the cracking moment is exceeded then the following model is used. The deflection of a beam subjected to cracking is made up of two components. The first contribution is due to an elastic deflection, while the second contribution is ascribed to cracking (figure 5). In order to obtain the cracking hinge rotation $\boldsymbol{\theta}$, the zone of cracking needs to be identified (figure 6).

It is assumed that all the cracking that occurs over this zone is lumped together to form one single crack at the position of maximum moment. The rotation that this "hinge" undergoes is equal to the integral of the curvatures across all cracks in the cracked zone.

In this cracked zone the concrete consists of cracked and uncracked sections. Allowance for the cracked portion of the cracked zone can be made by the factor ζ. Thus, the rotation of the hinge will be equal to the intergration of the curvature over this cracked length.

$$\boldsymbol{\theta} = \zeta \int_{x_2}^{x_2} \frac{M}{EI_{cr}} dx \ldots \quad eqn\ 5$$

This is similar to the development of the bilinear model used in the Manual on the CEB/FIP Model Code[2], except that in the Manual cracking was assumed to occur uniformly over the entire element length.

In the case of statically indeterminate horizontal slab strips with downward loading (i.e. propped cantilevers or beams built-in at both ends), only the zone in which the sagging moment exceeds the cracking moment needs to be considered. A cracking "hinge" at midspan will always be preceeded by a cracking hinge at the supports for normal uniform loading situations.

Once $\boldsymbol{\theta}$ is known, the cracking deflection δ can easily be determined from trigonometry (figure 7). The hinge is placed at the position of maximum moment. The distance x is therefore known. Thus

$$\zeta = x \tan \frac{\boldsymbol{\theta}(L - x)}{} \ldots \quad eqn\ 7$$

The deflections thus computed are only short-term (instantaneous). A computer program was developed incorporating these two models. The maximum predicted deflection for a series of slab configurations were computed and compared to experimental results.

ANALYSIS OF RESULTS

Figure 8 shows the slab configurations that were tested in the laboratory by undergraduate thesis students. The square slabs have dimensions 1.00 m by 1.00 m. by 0.04 m, while the rectangular slabs have dimensions 1.40 m by 1.00 m by 0.04 m. A yield line analysis was performed for each slab configuration to determine the design ultimate load. The design ultimate load was then divided by a factor of 1.6 to determine the design service load (the dead load was small at service loading). The maximum service deflection for each slab configuration was obtained from load-deflection curves.

The results of the computer program are shown in table 1. The first column in the table refers to the slab configuration as shown in figure 8. The second column refers to the experimentally observed deflection at the maximum service load for each particular slab. The next three columns (3, 4 & 5) refer in turn to the calculated elastic deflection, the deflection due to cracking and the total deflection for a strip spanning in the x-direction. The three columns thereafter (6,7 & 8) are similar, but for a strip spanning in the y-direction. The last two columns (9 & 10) show the average of the two total deflections and the ratio of this average total deflection to the experimentally observed deflection respectively.
The predicted maximum deflection is always higher than the experimentally observed maximum deflection. The ratio of predicted to observed deflection varies from 2.4 to 3.3 with an average value of 2.95 for the square slabs and from 2.2 to 3.7 with an average of 2.94 for the rectangular slabs.

Three phenomena are not included in the proposed models. These are:

i) The Poisson effect, where curvature about one axis will cause secondary reverse (anticlastic) curvature about the orthogonal axis.

ii) The surface-shearing action. The slab is very stiff in-plane and this action of the edge strips will prevent the centre of the slab from expanding to permit cracking.

iii) The membrane action caused by the supports restraining the slab from extending or shortening readily in-plane due to curvature. (This action was not present in the slabs tested in the laboratory as they were supported on rubber supports which are very flexible in shear).

These three phenomenon are not easily modelled and their effects thus cannot be easily quantified. They will cause the observed deflections to be lower than the predicted deflections, but the exact extent cannot be determined.

For the square slabs (1 to 6) the computed deflections using the proposed model for the x- and y-directions (columns 5 and 8) generally compare very well, except for slab 3. An interesting phenomenon is observed with slabs 2, 3 & 5. In each of these cases the strip spanning in the x-direction is not as rigidly supported as the strip spanning in the y-direction. As a result of this, the strip spanning in the y-direction attracts more of the load than the strip in the x-direction, as is expected. This is confirmed by the maximum elastic deflection in the y-direction being higher than that in the x-direction (column 6 vs.3). However, the deflection due to cracking is much higher for the strip less rigidly supported than the more rigidly supported strip! This phenomenon is also observed for the rectangular slabs (7 to 15), with the exception of slab 14, and is explained below.

The model developed for deflection due to cracking is based on the relationship

$$x_2 \quad M$$
$$x_2 \quad EI_{cr}$$

(Refer to figure 6 for the definition of x_1 & x_2). For two strips of equal length and stiffness and the same maximum moment in sagging, the strip that is less rigidly supported will have a longer cracked length (x_2-x_1). The factor ζ will be the same for both cases, so consequently the strip less rigidly supported will give a higher $\boldsymbol{\theta}$ and thus a higher deflection due to cracking. If on the other hand, the two strips of equal length and stiffness have the same cracked lengths, then the less rigidy supported slab will have a lower maximum moment in sagging. The value of the integral will therefore be lower. In addition, the factor ζ is dependent on M_m.

$$M_m = \sqrt{M_{max} \cdot M_{cr}}$$

$$\zeta = 1 - \beta_1\beta_2 \frac{M_{cr}^2}{M_m^2}$$

$$\therefore \zeta = 1 - \beta_1\beta_2 \frac{M_{cr}}{M_{max}}$$

Therefore, the lower the value of M_{max} (for sagging) the lower the value of ζ. This will lead to a smaller $\boldsymbol{\theta}$ and thus a lower deflection due to cracking.

Depending on which of the two cases described is dominant in each individual slab, it is possible for the less rigidly supported strip to attract less load, but to have a higher deflection due to cracking. This is the case for the majority of the slabs tested. A solution to this problem is to redefine M_m. If M_m is taken as some lower value than the CEB definition, then the contribution of the deflection due to cracking will be less. In addition to this, M_m must be defined as value that will not decrease so much as the degree of fixity of the supports increases. For a strip carrying a fixed load, the absolute value of the maximum bending moment in sagging, with a change of support fixity. The cracking moment capacity will also have to be included in the definition since it is a measure of the beam's ability to withstand concrete flexural tensile stresses. It is suggested to take a fraction of the geometric mean of the maximum moment (whether hogging or sagging) of the strip and its cracking moment capacity in sagging.

$$\text{e.g. } M_m = \frac{1}{2}\sqrt{M_{abs.max} \cdot M_{cr}}$$

CONCLUSIONS

The first computational model predicts the load dispersion at only nine points on the slab. If the slab is divided into a greater number of orthogonal strips with more intersection points then a more accurate load dispersion can be found. The continuity between strips spanning in the same direction will be vastly improved with the addition of extra strips in that direction. The inclusion of torsion in this model would also be a significant improvement, but with more strips this complicated refinement is deemed unnecessary.

With the second computational model, the coefficient ζ determines the contribution of the tension stiffening effect of the concrete to the overall deflection. This coefficient needs improving for the proposed model. The coefficient is proportional to M_m which should be redefined as a fraction (eg. 1/2) of the geometric mean of the maximum moment of the strip (whether hogging or sagging) and the cracking moment capacity in sagging.

The two proposed models do not predict deflections accurately enough. With the above improvements it will be a useful design aid, but only if reliable experimental results were available, so that a more accurate ratio of computed to actual deflection can be obtained. It will quantitatively express the effect of the surface-shearing action and the Poisson effect. This ratio can then be incorporated in the design process as a factor for appropriately reducing the computed deflections.

ACKNOWLEDGEMENTS

The author would like to acknowledge:

- Assoc. Prof. M.O. de Kock for his supervision of the thesis leading to this paper. His assistance with the final draught of this paper is also appreciated.
- Assoc. Prof. R.D. Kratz for his assistance with the development of the two computational models.

REFERENCES

1. American concrete institute "Building Code Requirements for Reinforced Concrete", (ACI 318M - 83), American Concrete Institute, June 1984.
2. Ceb manual "Cracking and Deformation", Comité Euro-International du Beton (CEB), Swiss Federal Institute of Technology, CH 1015, Lausanne, 1985.
3. Branson, D.E. "Deformation of Concrete Structures", Mc Graw Hill, USA, 1977.

SYNOPSIS

A review of Codes of Practice shows that deflections of two-way spanning slabs are treated superficially. A method is proposed that determines the allocation of a uniformly distributed load to a number of points on two-way spanning, edge-supported slabs. The slab is divided into a set of orthogonal strips and deflection equations are solved in terms of the loading. Once the load distribution is known, only the strip spanning through the maximum region of slab deflection, in each orthogonal direction, is considered. The total deflection is composed of an elastic deflection and a deflection due to cracking. The elastic deflection is obtained using the uncracked moment of inertia. The deflection due to cracking is obtained from integrating the curvatures over a "cracked zone", using the cracked moment of inertia. A factor is introduced to include the tension stiffening effect of the concrete. A computer program incorporating these two models is developed. The results show that the method over-predicts deflections. Future improvements include an increase of the number of strips that the slab is divided into and a reduc-

Table 1 - Results of Computer Program

Slab No (1)	Exp. Defl. x (mm) (2)	Elas. Defl. x (mm) (3)	Crack Defl. x (mm) (4)	Total Defl. y (mm) (5)	Elas. Defl. y (mm) (6)	Crack Defl. y (mm) (7)	Total Defl. y (mm) (8)	Ave. Defl. (mm) (9)	Ave ÷ Exp. (10)
1	1.6	0.9	4.3	5.2	0.9	4.3	5.2	5.2	3.3
2	-	0.8	3.8	4.7	1.1	3.1	4.2	4.4	-
3	3.0	1.1	9.8	10.9	1.8	4.1	5.9	8.4	2.8
4	2.7	1.3	5.2	6.5	1.3	5.2	6.5	6.5	2.4
5	2.3	1.6	8.0	9.6	1.7	4.1	5.8	7.7	3.3
6	-	1.9	5.3	7.2	1.9	5.3	7.2	7.2	-
7	-	1.5	8.8	10.3	1.2	11.0	12.2	11.2	-
8	4.9	2.4	9.0	11.4	1.4	12.9	14.3	12.8	2.6
9	-	1.4	8.1	9.5	1.6	7.4	9.0	9.2	-
10	-	1.6	12.3	13.9	1.9	4.3	6.3	10.1	-
11	5.2	2.7	2.3	5.0	1.6	16.0	17.6	11.3	2.2
12	3.8	2.6	9.8	12.4	1.7	8.2	9.9	11.2	2.9
13	-	2.6	1.1	3.7	2.1	11.8	14.0	8.8	-
14	2.9	2.6	10.4	13.0	2.3	6.1	8.3	10.7	3.7
15	2.4	2.9	2.5	5.5	2.5	7.9	10.4	7.9	3.3